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ANALYSIS OF APPROACHES TO EFFECTIVE DEGREES OF FREEDOM ESTIMATION IN THE PRESENCE OF OBSERVED CORRELATION BETWEEN THE RESULTS OF MEASUREMENT OF INPUT QUANTITIES

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One of the main principles of evaluating expanded uncertainty in the Guide to the Expression of Uncertainty in Measurement (GUM) [1] in the presence of significant contributions of type A uncertainty is to assign to the measurand the t-distribution with degrees of freedom ν_{eff} determined by the Welch-Satterthwaite equation.

In works [2,3] it is shown that for correlated data, this formula under certain, quite real conditions, leads to a value of the ν_{eff} equal to zero, which corresponds to an impossible value of expanded uncertainty equal to infinity.

The expressions for calculating ν_{eff} in the presence of an observed correlation between the results of measurements of input quantities, published in works [4,5], are analyzed.

For the analysis, a simple mathematical model is used in the form

$$Y = f(X_1, X_2) \tag{1}$$

of a linearizable function of two input quantities X_1, X_2 measured simultaneously n times, the measurement results of which are correlated with the correlation coefficient $r_{1,2}$ and contain only type A uncertainties with the same degrees of freedom $\nu = n-1$.

In the article by R. Willink [5] Result 1 is given for this case: "If any group of input quantities is estimated from a set of n repeated observations then this group of quantities should be combined to obtain a single component of uncertainty with *n-1* degrees of freedom". This Result 1 is confirmed by implementing the reduction method described in [4].

The article [4] gives a general formula (26) for the effective degrees of freedom for a model with N pairwise correlated input quantities with correlation coefficients r_{ij} and numbers of degrees of freedom ν_{ij}:

$$\nu_{eff} = \frac{u_y^4}{\sum_{i=1}^{N}\left(\sum_{j=1}^{N} c_i c_j r_{ij} u_i u_j\right)^2 \Big/ \nu_i}, \tag{2}$$

in which u_i, u_j and c_i, c_j are their standard uncertainties and the corresponding sensitivity coefficients.

Applying expression (2) to equation (1) yields the following result:

$$\nu_{eff} = \frac{(n-1)(c_1^2 u_{A1}^2 + 2r_{12}c_1c_2u_{A1}u_{A2} + c_2^2u_{A2}^2)^2}{(c_1^2u_{A1}^2 + r_{12}c_1c_2u_{A1}u_{A2})^2 + (r_{12}c_1c_2u_{A1}u_{A2} + c_2^2u_{A2}^2)^2}, \tag{3}$$

which is inconsistent with Result 1 of article [4].

In article [5] a general formula (46) for ν_{eff} is also given, similar to (2), in the form:

$$\nu_{eff} \cong \frac{u_y^4}{\sum_{i=1}^{N}\frac{c_i^4 u_i^4}{\nu_i} + \sum_{i=1}^{N-1}\sum_{j>i}^{N} r_{ij}^2 c_i^2 c_j^2 \left(\frac{u_i^2}{\nu_i}\right)\left(\frac{u_j^2}{\nu_j}\right)\left(\nu_i + \nu_j + \frac{1}{2}\right) + 2\sum_{i=1}^{N-1}\sum_{j>i}^{N} r_{ij} c_i c_j u_i u_j \left(\frac{c_i^2 u_i^2}{\nu_i} + \frac{c_j^2 u_j^2}{\nu_j}\right)}. \quad (4)$$

Applying expression (4) to equation (1) yields the following result:

$$\nu_{eff} \cong \frac{(n-1)(c_1^2 u_{A1}^2 + 2r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2}{c_1^4 u_1^4 + c_2^4 u_2^4 + r_{1,2}^2 c_1^2 c_2^2 u_1^2 u_2^2 \frac{2n-1{,}5}{(n-1)} + 2r_{1,2} c_1 c_2 u_1 u_2 (c_1^2 u_1^2 + c_2^2 u_2^2)}, \quad (5)$$

which is inconsistent with Result 1 of article [4].

The report discusses an extension of the Welch-Satterthwaite equation to correlated measurements, which allows avoiding the above effects. The equation proposed in [6] is an implementation of this extension:

$$\nu_{eff} = (n-1)\frac{u^4(y)}{\left[\sum_{i=1}^{N} u_{Ai}^2(y) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} r_{i,j} u_{Aj}(y) u_{Aj}(y)\right]^2}. \quad (6)$$

Applying expression (6) to equation (1) yields the following result:

$$\nu_{eff} = \frac{(n-1)(c_1^2 u_{A1}^2 + 2r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2}{(c_1^2 u_{A1}^2 + 2r_{12} c_1 c_2 u_{A1} u_{A2} + c_2^2 u_{A2}^2)^2} = n-1, \quad (7)$$

which is consistent with Result 1 of article [4].

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