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DISCRETE MARKOV CHAINS IN EXCHANGE MODELS

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The paper considers a stochastic approach to the analysis of discrete linear exchange models. Discrete linear exchange models are described by Markov chains, using Z-transformation. On the basis of this approach, discrete models of deficit-free, deficit and generalized (mxn size) exchange models are obtained. A two-level balance exchange model was created with a different number of participants at each level. At these levels, participants in vertically adjacent channels of distribution of goods in supply chains can be located: producers-consumers, distribution intermediaries. When the volume of supplies changes in proportion to the equilibrium distribution at the upper level, the ratio of the amount of benefits received at the lower level will remain. In this case, each recipient of the lower level knows his share of the distributed goods, which contributes to a more efficient organization of the exchange process. The technique used is of interest for creating multilevel models of distribution channels.

Introduction

Human entrepreneurial activity is always associated with the exchange of various goods (values). The beginning of the exchange process was trade, both within the country and between states. Currently, it is an important component of the socio-economic development of countries and plays a significant role in their integration into the international community. Any business entity deals with the planning of commodity-cash flows. The description of complex dynamic processes involves the use of mathematical methods, on which the quality of the constructed models depends. In this regard, the improvement of exchange models is of great interest for planning the economic activities of enterprises and the country's economy as a whole, and their further development and the creation of more improved models is an urgent task.

The main issues in exchange models are the issues of determining the equilibrium state, the time (number of steps) for which the system comes to an equilibrium state and the study of the dynamics of the system's transition from state to state that leads to it.

One of the approaches to studying exchange models is based on solving recurrent systems of linear equations, linear differential equations through finding the eigenvalues and eigenvectors [1-8].

The next approach is based on the study of stochastic models [9-15].

One of the directions of the stochastic approach to the study of exchange processes was developed on the basis of discrete Markov chains [16-20].

This approach makes it possible not only to determine the equilibrium states of the system, but also to observe the dynamics of the system's displacement from the initial state to the equilibrium state.

The next step was the development of continuous deficit-free and scarce models, both without control and with external control, building a multi-level balance exchange model with a different number of participants at each level.

1. Deficit-free and deficit exchange models

One of the ways to construct a discrete deficit-free linear exchange model (linear model of international trade) was Leontiev's balance model "Input - Output" [21]:

$$A\overline{X} + \overline{Y} = \overline{X}, \tag{1.1}$$

where $\overline{X}, \overline{Y}$ –the columns vectors of size *nx1*.

If $\overline{Y} = \overline{0}$, then equation (1.1) takes the form (1.2):

$$A\overline{X} = \overline{X} \tag{1.2}$$

and is a linear exchange model. Matrix A was called the structural trading matrix, which has the following properties:

1. All elements of this matrix are non-negative ($\forall i, j, a_{ij} \geq 0$);
2. The sum of all elements of each column of matrix A are equal to 1 ($\sum_{i=1}^{n} a_{ij} = 1, \forall j = \overline{1,n}$).

It is clear that if the vector $\overline{X}$ is normalized, then the solution (1.2) will be the eigenvector of the matrix A for $\lambda = 1$.

It is easy to see that the transposed matrix A satisfies the conditions:

$$L = A^T; 0 \le a_{ij} \le 1, \sum_{j=1}^{n} a_{ji} = 1, \forall i = \overline{1, n}, \tag{1.3}$$

then the transposed equation (1.2) takes the form:

$$\overline{p}L = \overline{p} \tag{1.4}$$

where $\overline{p} = (\overline{X^0})^T = (p_1, \quad p_2, \quad ..., \quad p_n)$, $\left\| \overline{X^0} \right\| = \sum_{i=1}^{n} p_i = 1$.

This means that the matrix L is a stochastic transition matrix (in one step) from state *i* to state *j* of the Markov chain. To determine the Markov chain, it is also necessary to specify some initial distribution of the random variable $\overline{\xi} = \overline{p}(0)$ Equation (1.4) after *n* steps takes the form:

$$\overline{p}(n) = \overline{p}(n-1)L = ... = \overline{p}(0)L^n. \tag{1.5}$$

Applying the Z-transformation to equation (1.5), we obtain:

$$z^{-1}(H(z) - \overline{p}(0)) = H(z)L, \tag{1.6}$$

where $\overline{p}(n) \leftrightarrow H(z)$, from which it follows that $H(z) = \overline{p}(0)(I - zL)^{-1}$, where I is an identity matrix, and $(I - zL)^{-1} \leftrightarrow L^n$. Thus, expanding the matrix $(I - zL)^{-1}$ into prime factors, we obtain an analytical solution to the problem.

Note 1. The matrix L^n can always be represented as a sum of two summonds:

$$L^n = S + T(n) \tag{1.7}$$

Note 2. The matrix S for an ergodic process is always a stochastic matrix, each row of which contains the final distributions of the Markov chain. This matrix is called stationary because it does not depend on n.

Note 3. The matrix S coincides with the matrix $L(\infty) \to L^n, n \to \infty$.

Note 4. For an ergodic process, the matrices *T(n)* describe the behavior of the process during the transition period. They are matrices multiplied by coefficients of the form $\alpha^n, n\alpha^n, ...$ and so on. Naturally, these terms are infinitesimal with increasing *n*, since they correspond to a decreasing geometric progression with $|\alpha|<1$. These matrices are also interesting in that the sum of their elements for each row is equal to zero. Matrices with such properties are called differential.

2. Deficit exchange model

Now let's go to the description of the deficit model of exchange between *n* partners.

Let x_i is a total supply of goods (values) of exchange possessed by the *i*-th exchange partner; x'_j is a total demand for goods (values) of the *j*-th exchange partner.

In this case

$$x_i = x_{i1} + x_{i2} + \cdots + x_{in} = \sum_{k=1}^{n} x_{ik}, \quad i = \overline{1,n}; \tag{2.1}$$

$$x'_j = x_{i1} + x_{i2} + \cdots + x_{in} = \sum_{k=1}^{n} x_{kj}, \quad j = \overline{1,n}. \tag{2.2}$$

Equation (2.1) is the supply equation, and (2.2) is the demand equation, and it is clear from the problem statement that

$$x_{ij} \geq 0, x_i > 0, x_j > 0.$$

The total consolidated budget D of these n partners will be equal to:

$$D = \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x'_j . \tag{2.3}$$

The matrix $X = (x_{ij}), (i, j = \overline{1,n})$ represents the distribution of budgets between n exchange partners and has the form:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}. \tag{2.4}$$

Let

$$a'_{ij} = \frac{x_{ij}}{x'_j}, (i, j = \overline{1,n}) \tag{2.5}$$

is a part of the budget that the j-th exchange partner spends on the purchase of goods (values) from the i-th partner, and

$$a_{ij} = \frac{x_{ij}}{x_i}, (i, j = \overline{1, n}) \tag{2.6}$$

is a part of the budget that the i-th partner receives in the form of goods (values) from the j-th partner.

Let's compose a matrix A_1 from the coefficients (2.5):

$$A_1 = \begin{pmatrix} \frac{x_{11}}{x'_1} & \frac{x_{12}}{x'_2} & \dots & \dots & \frac{x_{1n}}{x'_n} \\ \frac{x_{21}}{x'_1} & \frac{x_{22}}{x'_2} & \dots & \dots & \frac{x_{2n}}{x'_2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{x_{n1}}{x'_1} & \frac{x_{n2}}{x'_2} & \dots & \dots & \frac{x_{nn}}{x'_n} \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & \dots & \dots & a'_{1n} \\ a'_{21} & a'_{22} & \dots & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a'_{n1} & a'_{n2} & \dots & \dots & a'_{nn} \end{pmatrix}. \tag{2.7}$$

The transposed matrix A_1^T is called the transition matrix from the supply vector to the demand vector.

The matrix A_2 composed of the coefficients (2.6) is called the transition matrix from the demand vector to the supply vector:

$$A_2 = \begin{pmatrix} \frac{x_{11}}{x_1} & \frac{x_{12}}{x_1} & \dots & \dots & \frac{x_{1n}}{x_1} \\ \frac{x_{21}}{x_2} & \frac{x_{22}}{x_2} & \dots & \dots & \frac{x_{2n}}{x_2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{x_{n1}}{x_n} & \frac{x_{n2}}{x_n} & \dots & \dots & \frac{x_{nn}}{x_n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{22} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}. \tag{2.8}$$

From the construction of matrices (2.7) - (2.8) follows that the sum of the elements of the matrix A_1 in each column is 1, and the matrix A_2 has the sum of the elements in each row equal to 1.

Note that in order the exchange between partners to be balanced and deficit-free, the following equalities must be met:

$$x_i = x'_j, \quad i = j, \quad (i, j = \overline{1, n}). \tag{2.9}$$

Using matrices (2.7) - (2.8), we obtain the matrix equations:

$$\overline{p}A_2 = \overline{q}\,, \tag{2.10}$$

$$\overline{q}A_1^T = \overline{p}\,, \tag{2.11}$$

where $\overline{p} = (p_1, p_2, \cdots, p_n) = \left(\frac{x_1}{D}, \ \frac{x_2}{D}, \ \ldots \ \frac{x_n}{D} \right)$ and

$$\overline{q} = (q_1, q_2, \cdots, q_n) = \left(\frac{x_1^{'}}{D}, \ \frac{x_2^{'}}{D}, \ \ldots \ \frac{x_n^{'}}{D} \right)$$

determine the vectors of states of distribution of goods (values) between partners depending on supply and demand, respectively.

Substituting in (2.10) instead of $\overline{p}$ its expression from (2.11), and instead of $\overline{q}$ its expression from (2.10) into expression (2.11), we obtain:

$$\begin{cases} \overline{p} \quad A_2 A_1^T = \overline{p} \\ \sum_{k=1}^{n} p_k = 1 \end{cases}; \tag{2.12}$$

$$\begin{cases} \overline{q} \quad A_1^T A_2 = \overline{q} \\ \sum_{k=1}^{n} q_k = 1 \end{cases}. \tag{2.13}$$

The matrices $A_2\ A_1^T$ and $A_1^T\ A_2$ will be called the supply and demand matrices, respectively. These matrices are stochastic and under the initial conditions $\overline{p}(0)$ and $\overline{q}(0)$ form Markov chains for supply and demand models.

Note that in the case of a balanced deficit-free exchange, the limiting states of the matrices $A_2\ A_1^T$ and $A_1^T\ A_2$ coincide ($\overline{p^*} = \overline{q^*}$). Thus, to study the behavior of a balanced deficit-free exchange, it is enough for us to have either the exchange supply matrix $A_2\ A_1^T$ or the exchange demand matrix $A_1^T\ A_2$.

Note 5. Let's add a vector $\overline{f}$ to equation (1.5), then we get:

$$\overline{p}(n) = \overline{p}(n-1)L + \overline{f}\,. \tag{2.14}$$

The vector $\overline{f}$ will be considered as a vector of administrative redistribution of goods (values). For the redistribution process to converge, the condition should be met: $\sum_{i=1}^{n} f_i = 0$.

To find an analytical solution to equation (2.14), we apply the Z-transformation.

3. Two-level exchange model

Consider a direct supply chain (a simple distribution channel of goods (values)): "sellers (producers)" - "buyers (consumers)" [22]. Let P_j^0 the number of "sellers" be at the top zero level ($j = \overline{1, m_0}$), and the number of "buyers" at the first P_i^1 level ($i = \overline{1, m_1}$). Each "seller" distributes the goods (values) between the "buyers" in accordance with contracts that take into account the possibility of changing the supply volumes while maintaining shares between them. "Buyers" exchange goods (values), carrying out a flow of exchange to "sellers". Matrices of paired comparisons of the volumes of goods (values) in relative terms from "buyers" to each "seller" are presented in table 3.1.

Table 3.1.

Pairwise comparison matrices for level 1 participants relative to each level 0 participant

P_j^0	P_1^1	P_2^1	P_3^1	$\dots$	$P_{m_1}^1$	Sum	$V(P_j^0, P^1)$
P_1^1	1	x_{1j}/x_{2j}	x_{1j}/x_{3j}	$\dots$	$x_{1j}/x_{m_1 j}$	$x_{1j} * t_j$	x_{1j}/X_j^0
P_2^1	x_{2j}/x_{1j}	1	x_{2j}/x_{3j}	$\dots$	$x_{2j}/x_{m_1 j}$	$x_{2j} * t_j$	x_{2j}/X_j^0
P_3^1	x_{3j}/x_{1j}	x_{3j}/x_{2j}	1	$\dots$	$x_{3j}/x_{m_1 j}$	$x_{3j} * t_j$	x_{3j}/X_j^0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$P_{m_1}^1$	$x_{m_1 j}/x_{1j}$	$x_{m_1 j}/x_{2j}$	$x_{m_1 j}/x_{3j}$	$\dots$	1	$x_{m_1 j} * t_j$	$x_{m_1 j}/X_j^0$
$t_j = \frac{1}{x_{1j}} + \frac{1}{x_{2j}} + \frac{}{x_{3j}} + ... + \frac{1}{x_{m_1 j}}$ $X_j^0 = x_{1j} + x_{2j} + x_{3j} + ... x_{m_1 j}$					Sum	$X_j^0 * t_j$	1

In the Table 3.1 x_{ij} is the amount of goods that the i-th "buyer" transfers to the j-th "seller" or vice versa (the j-th "seller" transfers the goods (values) of exchange to the i-th "buyer"), $V(P_j^0, P^1)$ are eigenvectors of the matrices P_j^0. For all tables of distribution of exchange between participants of 0 and 1 levels, the following equalities are fulfilled:

$$\sum_{i=1}^{m_1} x_{ij} = X_j^0, \quad \sum_{j=1}^{m_0} x_{ij} = X_i^1, \quad \sum_{j=1}^{m_0} X_j^0 = \sum_{i=1}^{m_1} X_i^1 = D$$

Matrices of paired comparisons of the received goods (values) between the "sellers" in relation to each "buyer" are presented in table 3.2.

Table 3.2

Pairwise comparison matrices for level 0 participants relative to each level 1 participant

P_i^1	P_1^0	P_2^0	P_3^0	$\dots$	$P_{m_0}^0$	Sum	$V(P_i^1, P^0)$
P_1^0	1	$\frac{x_{i1}}{x_{i2}}$	$\frac{x_{i1}}{x_{i3}}$	$\dots$	$\frac{x_{i1}}{x_{im_0}}$	$x_{i1} * t_i$	$\frac{x_{i1}}{X_i^1}$
P_2^0	$\frac{x_{i2}}{x_{i1}}$	1	$\frac{x_{i2}}{x_{i3}}$	$\dots$	$\frac{x_{i2}}{x_{im_0}}$	$x_{i2} * t_i$	$\frac{x_{i2}}{X_i^1}$
P_3^0	$\frac{x_{i3}}{x_{i1}}$	$\frac{x_{i3}}{x_{i2}}$	1	$\dots$	$\frac{x_{i3}}{x_{im_0}}$	$x_{i3} * t_i$	$\frac{x_{i3}}{X_i^1}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$P_{m_0}^0$	$\frac{x_{im_0}}{x_{i1}}$	$\frac{x_{im_0}}{x_{i2}}$	$\frac{x_{im_0}}{x_{i3}}$	$\dots$	1	$x_{im_0} * t_i$	$\frac{x_{im_0}}{X_i^1}$
$t_i = \frac{1}{x_{i1}} + \frac{1}{x_{i2}} + \frac{1}{x_{i3}} + \dots + \frac{1}{x_{im_0}}$ $X_i^1 = x_{i1} + x_{i2} + x_{i3} + \dots + x_{im_0}$					Sum	$X_i^1 * t_i$	1

Here $V(P_i^1, P^0)$ are eigenvectors of matrices P_i^1.

Using the found eigenvectors $V(P_j^0, P^1) \quad (j = \overline{1, m_0})$ and $V(P_i^1, P^0) \quad (i = \overline{1, m_1})$ construct matrices $V(P^0, P^1)$ and $V(P^1, P^0)$:

$$V(P^0,P^1)=\begin{pmatrix} V^T(P_1^0,P^1) \\ V^T(P_2^0,P^1) \\ V^T(P_3^0,P^1) \\ \dots \\ V^T(P_{m_0}^0,P^1) \end{pmatrix}; \tag{3.1}$$

$$V(P^1,P^0)=\begin{pmatrix} V^T(P_1^1,P^0) \\ V^T(P_2^1,P^0) \\ V^T(P_3^1,P^0) \\ \dots \\ V^T(P_{m_1}^1,P^0) \end{pmatrix}. \tag{3.2}$$

The stochastic matrix $C(P^0)$ size $m_0 * m_0$ is calculated by the formula:

$$C(P^0)=V(P^0,P^1)*V(P^1,P^0). \tag{3.3}$$

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$$C(P^1)=V(P^1,P^0)*V(P^0,P^1). \tag{3.4}$$

The vector of global priorities $\overline{W}(P^0)$ is found from the system of equations:

$$\begin{cases} \overline{W}(P^0)=\overline{W}(P^0)*C(P^0) \\ \sum_{j=1}^{m_0} w_j(P^0)=1 \\ \overline{W}(P^0)=\left(w_1(P^0), \quad w_2(P^0), \quad w_3(P^0), \quad \dots \quad ,w_{m_0}(P^0)\right) \end{cases}. \tag{3.5}$$

The vector of global priorities $\overline{W}(P^1)$ is found from the system of equations:

$$\begin{cases} \overline{W}(P^1)=\overline{W}(P^1)*C(P^1) \\ \sum_{j=1}^{m_1} w_j(P^1)=1 \\ \overline{W}(P^1)=\left(w_1(P^1), \quad w_2(P^1), \quad w_3(P^1), \quad \dots \quad ,w_{m_1}(P^1)\right) \end{cases}. \tag{3.6}$$

The vectors of global priorities $\overline{W}(P^0)$, $\overline{W}(P^1)$ determine the equilibrium points of the systems of distribution of goods (values) in the two-level balance model at each level.

Conclusions

The paper considers a stochastic approach to the analysis of discrete linear exchange models: deficit-free, deficit, generalized with an external control vector, two-level balance exchange model. These models are described by Markov chains using Z-transformation, which allows to obtain an analytical form of solving problems. This technique allows to describe the models: international trade, the implementation of budget projects, financial interaction of enterprises in the regions, logistics supply chains and many other models.

A significant difference between the probabilistic approach is that there is no need to use the theory of eigenvalues and eigenvectors, since matrix expressions are simpler and that the fundamental matrix allows direct probabilistic interpretation, in contrast to eigenvalues.

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