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Normalized Difference Model of the Descriptor Control System

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Abstract

A difference model approximating a semi-linear descriptor control system with continuous time is studied to enable further application of the results in developing neural networks. An algorithm has been developed to transform the implicit difference equation so that, in its linear part, the original regular characteristic pencil of two matrices is converted into a normalized pencil with an identity matrix as the free term and a certain "informational" matrix T associated with the spectral parameter. One of the key steps in the algorithm involves reducing the matrix T to its Jordan form with a special arrangement of Jordan blocks. An example of a descriptor dynamic system in an electrical network is considered.

Keywords

neural network, control system, dynamic system, descriptor system, electrical network

The article examines a semi-linear descriptor dynamic control system. The analysis includes the original system and its evolution law, as well as a transformation to another system whose state and evolution significantly simplify the application of neural networks for predicting the states of the original system. As an example of application, a descriptor system in an electrical network is considered.

1. Normalized difference approximation

A semi-linear descriptor control system is considered, where the state vector function $x(t) \in \mathbb{R}^n, t \in [0, \theta)$ satisfies the implicit differential-algebraic equation

$$\frac{d}{dt}\big(A_0 x(t)\big) + B_0 x(t) = F\big(x(t)\big) + L_0 u(t)\,. \tag{1}$$

For the theory of differential-algebraic equations and their applications, we refer to [1,2]. In equation (1) $F: \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a nonlinear mapping, $u(t) \in \mathbb{R}^l (\forall t),\ L_0$ is a matrix of size $n \times l, \lambda A_0 + B_0$ is a regular pencil of $n \times n$-matrices, $A_0 \neq \mathbf{0}$. The finite spectrum of the pencil $\sigma_0 = \sigma(A_0, B_0) = \{\lambda_j\}$ consists of its eigenvalues λ_j — the roots of the characteristic polynomial $p_0(\lambda) = det(\lambda A_0 + B_0)$. All other points in the plane $\mathbb{C}$ form the set of regular points $\rho = \rho(A_0, B_0) = \mathbb{C} \setminus \sigma\,(A_0, B_0)$, at each of which the inverse matrix $(\lambda A_0 + B_0)^{-1},\ \lambda \in \rho$ exists.

For the difference approximation of equation (1) with a constant time step $\Delta > 0$, a set of isolated points $\{t_k\}$ is chosen in the interval $[0, \theta)$: $t_k = k \cdot \Delta,\ \ k = 0,1,2, \ldots$ Replacing the differentials in (1) with finite differences at points t_k and denoting $x_k = x(t_k), u_k = u(t_k)$, we obtain the difference equation:

$$\Delta^{-1} A_0 x_{k+1} + B_0 x_k - \Delta^{-1} A_0 x_k = F(x_k) + L_0 u_k. \tag{2}$$

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The time step $\Delta > 0$ can be chosen small enough such that $\Delta^{-1} > \max\{|\lambda_j|, \forall \lambda_j \in \sigma_0\}$. Then $\pm\Delta^{-1} \epsilon\, \rho(A_0, B_0)$. Rewrite equation (2) in the form

$$Ax_{k+1} + Bx_k - Ax_k = F(x_k) + L_0 u_k\,, \tag{3}$$

where $A = \Delta^{-1} A_0, B = B_0$. Neural networks [3] are proposed for solving the difference equation (3), which approximates the differential-algebraic equation (1).

The spectrum σ of the characteristic pencil $S(\lambda) = \lambda A + B$ lies inside the unit circle $|\lambda| < 1$. Therefore $\pm 1 \in \rho(A, B)$ and the inverse matrices $D = (B - A)^{-1},\ (B + A)^{-1}$ exist.
Equation (3) is equivalent to the following equation, normalized to the identity matrix E associated with x_k and the informational matrix $T = DA$ associated with x_{k+1}:

$$Tx_{k+1} + x_k = \Psi(x_k) + Lu_k, \tag{4}$$

where $\Psi(x_k) = DF(x_k), L = DL_0$.

2. Jordan form of the informational matrix *T*

If the matrix T is non-invertible, the permissible initial data x_0 in equation (4) cannot be arbitrary. They must satisfy a specific algebraic constraint linking the components of the vectors x_0, u_0. To derive this constraint, the Jordan form J of matrix T is employed. A Python program was developed to compute an invertible matrix Q that transforms T into its Jordan form with a specific ordering of Jordan blocks:

$$J = QTQ^{-1} = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix},\ T = Q^{-1}JQ. \tag{5}$$

The block-diagonal matrix G consists of invertible blocks J_i of dimensions $p_i \times p_i$, while matrix H consists of nilpotent blocks N_j of dimensions $s_j \times s_j$:

$$G = diag\{J_1, J_2, \dots, J_r\}, \sum_{i=1}^{r} p_i = m, \tag{6}$$

$$H = diag\{N_1, N_2, \dots, N_q\}, \textstyle\sum_{j=1}^{q} s_j = n - m, \tag{7}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 \cdots & 0 & 0 \\ 0 & \lambda_i & 1 \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \quad \cdots & \lambda_i & 1 \\ 0 & 0 & 0 \quad \cdots & 0 & \lambda_i \end{pmatrix},\quad N_j = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & 1 \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \cdots & 0 & 1 \\ 0 & 0 & 0 \cdots & 0 & 0 \end{pmatrix}. \tag{8}$$

The special ordering of the blocks is such that: $s_1 \geq s_2 \geq \cdots \geq s_q \geq 1$,

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0,\ |\lambda_i| = |\lambda_{i+1}| \Rightarrow p_i \geq p_{i+1},\ \ i \leq r - 1.$$

3. Analysis of the normalized model

Let us introduce projection $n \times n$-matrices V_i with elements v_{kj}^i $(i = 1,2, \dots, n)$ such that each matrix V_i has a single nonzero element $v_{ii}^i = 1$ on the main diagonal:

$$v_{kj}^i = \begin{cases} 1, k = j = i \\ 0, |k - i| + |j - i| \neq 0;\ k, j = 1,2, \dots, n \end{cases}$$

The action of the matrix V_i on an n-dimensional state vector preserves the component with index i and nullifies all other components.

Define the projection $n \times n$-matrices

$$P_G = \sum_{i=1}^{m} V_i, \quad P_H = \sum_{i=m+1}^{n} V_i. \tag{9}$$

Using Jordan form (5) equation (4) can be rewritten in the form of the following equation with respect to $y_k = Qx_k$:

$$Jy_{k+1} + y_k = Q\Psi(Q^{-1}y_k) + QLu_k. \tag{10}$$

With the help of projectors (9), equation (10) splits into two equations:

$$Gy_{k+1}^G + y_k^G = P_G Q\Psi(Q^{-1}y_k) + P_G QLu_k, \tag{11}$$

$$Hy_{k+1}^H + y_k^H = P_H Q\Psi(Q^{-1}y_k) + P_H QLu_k, \tag{12}$$

where $y_k^G = P_G y_k$, $y_k^H = P_H y_k$.

Due to the invertibility of the $m \times m$-matrix G equation (11) is transformed into an explicit difference expression for the vector y_{k+1}^G in terms of the full vector y_k at the previous time step k:

$$y_{k+1}^G = -G^{-1}y_k^G + G^{-1}P_G Q\Psi(Q^{-1}y_k) + G^{-1}P_G QLu_k. \tag{13}$$

To analyze equation (12), we construct a self-adjoint projection $n \times n$-matrix $P_0 = P_0^*$ onto the kernel of the matrix J^* such that $J^* P_0 = 0 \sim P_0 J = 0$.

From the structure of the nilpotent blocks N_j, it follows that

$$P_0 = \sum_{j=1}^{q} V_{\alpha_j}, \quad \alpha_j = m + s_1 + s_2 + \cdots + s_j. \tag{14}$$

Introducing the additional projection matrix $P_1 = P_H - P_0$, and applying the projectors P_1, P_0 to equation (12), we split (12) into two equations. It can be verified that $P_1 H = H$, $P_1 P_H = P_1$, $P_0 H = 0$, $P_0 P_H = P_0$. Consequently, equation (12) is equivalent to the equations:

$$P_1 H P_H y_{k+1} = -P_1 y_k + P_1 Q\Psi(Q^{-1}y_k) + P_1 QLu_k \tag{15}$$

$$0 = -P_0 y_k + P_0 Q\Psi(Q^{-1}y_k) + P_0 QLu_k. \tag{16}$$

For a given control u_k, the set of admissible n-dimensional vectors y_k, satisfying the algebraic relationship (16), forms a manifold $\Lambda_k = \{y_k\}$ in the space $\mathbb{R}^n$ or in the complex space $\mathbb{C}^n$ if there exist complex spectrum points λ_i of the matrix T. By construction, the matrices P_G, P_1, P_0 are mutually orthogonal projection matrices and $P_G + P_1 + P_0 = E$.

Let $y_k \in \Lambda_k$ be a known state. Is it possible, based on y_k and the controls u_k, u_{k+1} to obtain the state y_{k+1}, that satisfies equation (10)? It is sufficient to find the three projections in the decomposition of the vector y_{k+1}:

$$y_{k+1} = P_G y_{k+1} + P_1 y_{k+1} + P_0 y_{k+1}. \tag{17}$$

The first two projections are uniquely determined through the difference relations (13), (15). Let us represent the algebraic relation (16) for the moment *k+1* as:

$$P_0 y_{k+1} - P_0 Q\Psi(Q^{-1}y_{k+1}) = P_0 QLu_{k+1}. \tag{18}$$

In the right-hand side of equality (17), the first two terms at the moment *k+1* are replaced by their already known representations (13), (15) through the data from the preceding moment k, and the obtained representation for y_{k+1} is substituted into the term with nonlinearity Ψ in (18). As a result, an algebraic equation is obtained for the projection $P_0 y_{k+1}$.

If the original mapping F in (1), (2), (3) is linear, then Ψ (4) is also linear, and $P_0 y_{k+1}$ is uniquely determined from the relation (18).

For the nonlinear mapping Ψ, restrictions depending on the type of nonlinearity must be imposed. Under 'favorable' properties of the mapping Ψ the outlined procedure guarantees the existence and uniqueness of the desired projection $P_0 y_{k+1}$, and thus the existence of a recurrent mapping

$$\Phi_k\colon \Lambda_k \times \mathbb{R}^l \times \mathbb{R}^l \to \Lambda_{k+1}, \quad y_{k+1} = \Phi_k(y_k, u_k, u_{k+1}). \tag{19}$$

It follows from (19) and the relation $y_k = Qx_k$ that for the sequence of states $\{x_0, x_1, \dots, x_k, x_{k+1}, \dots\}$ of difference equation (2) or (3) or (4) there exist the recurrent mappings

$$x_{k+1} = M_k(y_k, u_k, u_{k+1}) = Q^{-1}\Phi_k(Qx_k, u_k, u_{k+1}). \tag{20}$$

Next, we will demonstrate how the proposed approach to the study of nonlinear differential-algebraic equations is applied to the controlled radio engineering system from Section 4. This approach can also be applied to other classes of radio engineering systems, for example, from works [4,5].

4. Example. Transient state equations of an electrical network

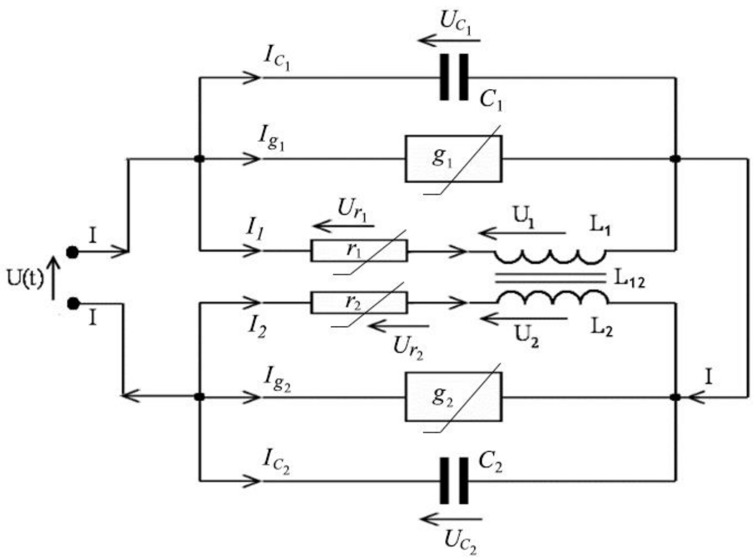

Figure 1. A two-terminal network with nonlinear resistances r_j and conductances $g_j, j = 1,2$.

The electrical network in Fig. 1 represents a two-terminal network with a given input voltage $U(t)$ on the pair of input terminals, two capacitances C_j, two nonlinear conductances g_j, inductances L_1, L_2, mutual inductance L_{12}, and two nonlinear resistances $r_j, j = 1,2$.

The currents, voltages, and element parameters of the network satisfy the relationships:

$$\left.\begin{aligned} I_{C_j} = C_j \frac{dU_{C_j}}{dt}, \quad I_{g_j} = g_j(U_{C_j}); \quad U_{r_j} = r_j(I_{r_j}) \\ U_1 = L_1\frac{dI_1}{dt} + L_{12}\frac{dI_2}{dt}; \quad L_j > 0, \quad L_{12} > 0 \\ U_2 = L_{12}\frac{dI_1}{dt} + L_2\frac{dI_2}{dt}; \quad L_{12}^2 = L_1 L_2; \quad L_1 \neq L_2 \end{aligned}\right\} \tag{21}$$

The currents through the inductances and *the voltages across the capacitors* are chosen as the dynamic state variables of the network. For the network in Fig. 1 the state vector $x(t)$ is four-dimensional:

$$x(t) = \left(I_1(t), I_2(t), U_{C_1}(t), U_{C_2}(t)\right)^{tr} \tag{22}$$

while the total number of currents and voltages across all nine branches of the circuit equals 18. In general, the complete system of Kirchhoff's equations can be written, for instance, using the fundamental cycle and cut-set matrices of the network graph. For the model network, this results in nine Kirchhoff equations. By eliminating the external current I, the currents I_{r_j}, and the voltages U_{g_j}, the following four conservation law equations are obtained ($j = 1,2$):

$$U_j - U_{C_j} = -U_{r_j}, U_{C_1} - U_{C_2} = U(t),\ I_{C_1} + I_{C_2} + I_1 + I_2 = -I_{g_1} - I_{g_2}. \tag{23}$$

Substituting the expressions for $U_j, I_{C_j}, U_{r_j}, I_{g_j}$ from (21), we obtain the desired system of differential-algebraic equations with respect to the dynamic variables (22):

$$\left.\begin{aligned}
&L_1 \frac{dI_1}{dt} + L_{12}\frac{dI_2}{dt} - U_{C_1} = -r_1(I_1)I_1 \\
&L_{12} \frac{dI_1}{dt} + L_2\frac{dI_2}{dt} - U_{C_2} = -r_2(I_2)I_2 \\
&U_{C_1} - U_{C_2} = U(t) \\
&C_1\frac{dU_{C_1}}{dt} + C_2\frac{dU_{C_2}}{dt} + I_1 + I_2 = -g_1(U_{C_1})U_{C_1} - g_2(U_{C_2})U_{C_2}
\end{aligned}\right\} \tag{24}$$

The system of differential-algebraic equations (24) takes the vector form (1) with respect to the state vector $x(t)$ (22), where:

$$A_0 = \begin{bmatrix} L_1 & L_{12} & 0 & 0 \\ L_{12} & L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & C_2 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, F(x) = \begin{bmatrix} -r_1(x_1)x_1 \\ -r_2(x_2)x_2 \\ 0 \\ -g_1(x_3)x_3 - g_2(x_4)x_4 \end{bmatrix}, L_0u = \begin{bmatrix} 0 \\ 0 \\ U(t) \\ 0 \end{bmatrix}.$$

Both matrices A_0, B_0 are singular: $det\, A_0 = 0,\ det\, B_0 = 0$. At the same time $rang A_0 = 2$ due to the parameter relationships in (21). The characteristic polynomial in this case is

$$p_0(\lambda) = det(\lambda A_0 + B_0) = a \cdot \lambda,\ a = \left(\sqrt{L_1} - \sqrt{L_2}\right)^2. \tag{25}$$

Thus, under the condition $L_1 \neq L_2$ the pencil $\lambda A_0 + B_0$ is regular with a single eigenvalue of 0, and for any $\lambda \neq 0$ the inverse matrix $(\lambda A_0 + B_0)^{-1}$ exists.

To illustrate, let us assign numerical values to the parameters of the circuit's inertial elements—inductances and capacitances:

$$L_1 = 4, L_{12} = 2, L_2 = 1, C_1 = 1, C_2 = 2.$$

The matrix $S = B_0 - A_0$ is invertible, with the inverse $S^{-1} = D$. Since $\pm 1 \in \rho(A_0 B_0)$, it is convenient to choose a time step $\Delta = 1$ when discretizing equation (1). Thus, the difference model of type (2) for equation (1) immediately takes the form (3), where $A = A_0, B = B_0$. The matrices $D = S^{-1}, T$ in the normalized form (4) of equation (1) are expressed as follows:

$$D = \begin{bmatrix} -4 & 7 & -5 & -1 \\ 7 & -13 & 9 & 2 \\ 1 & -2 & 2 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} -2 & -1 & -1 & -2 \\ 2 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Calculations using our Python program yielded the following results for transforming the matrix T into its Jordan form $J = QTQ^{-1}$:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Q^{-1} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The specific order of Jordan block arrangement in (6), (7) is preserved:

$$G = J_1 = -1, N_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, s_1 = 2, N_2 = 0, s_2 = 1.$$

Thanks to this, the required projectors take the form of the following diagonal matrices:

$$P_G = P_{J_1} = diag\{1\ 0\ 0\ 0\}, P_H = diag\{0\ 1\ 1\ 1\}, P_0 = diag\{0\ 0\ 1\ 1\}, P_1 = diag\{0\ 1\ 0\ 0\}.$$

Conclusion

The traditional approach to studying the descriptor system described by a linear or semi-linear differential-algebraic equation depends on the specific fixed index of the characteristic matrix pencil of the linear part of the equation (1). A key feature and versatility of our method is its independence from this specified index. It is also worth noting the potential use of the method in problems of conflict control and pursuit games.

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